The Heat Equation and Preliminaries

Boundary Value Problems, Eigenfunctions, Fourier Series, and the Heat Equation

Objectives

To practice solving boundary value problems, finding eigenfunctions of a differential operator, and expanding functions in a Fourier series. To better understand how these are all necessary components of solving the Heat Equation.

Recitation Worksheet Problems: Sections 7.1, 7.2, 7.3.

1. Consider the differential equation $y'' + 4y = 0$. Find an example of boundary conditions of the form ($y(a) = y_0$ and $y(b) = y_1$) and an example of the form ($y(c) = y_2$ and $y'(d) = y_3$), such that the corresponding boundary value problem

(a) has a unique solution;

(b) has no solution;

(c) has infinitely many solutions.

2. Find the eigenvalues and eigenfunctions for the differential operator $L(y) = -y''$ with boundary conditions $y(0) = 0$ and $y(3) = 0$.

3. Consider the function $f(x) = 2x + 1$ defined on $(0, 3]$ and extend it to an **odd** function $f_o(x)$ defined on $[-3, 3]$. Find the Fourier series expansion of $f_o(x)$.

4. Consider the function $f(x) = 2x + 1$ defined on $(0, 3]$ and extend it to an **even** function $f_e(x)$ defined on $[-3, 3]$. Find the Fourier series expansion of $f_e(x)$.

5. Consider the conduction of heat in a 3 cm long rod with conductivity constant $k = 1$, whose ends are maintained at 0°C for all $t > 0$. Find an expression for the temperature $u(t, x)$ if the initial temperature distribution in the rod is given by $u(0, x) = 2x + 1$.
Hint: Make use of the work you did in problems 2 and 3.

Answers

1. *Answers may vary.* Unique solution: $y(0) = 0, y(\pi/4) = 5$; No solution: $y(0) = 0, y(\pi/2) = 1$; Infinitely many solutions: $y(0) = 0, y(\pi/2) = 0$.

2. $\lambda_n = \left(\frac{n\pi}{3}\right)^2$, $y_n = \sin\left(\frac{n\pi}{3}x\right)$, $n = 1, 2, 3, \ldots$

3. $f_o(x) = \begin{cases} 2x-1, & x \in [-3,0) \\ 0, & x = 0 \\ 2x+1, & x \in (0,3] \end{cases}$, $\quad F[f_o](x) = \sum_{n=1}^{\infty} \left(-\frac{14}{n\pi}(-1)^n + \frac{2}{n\pi}\right) \sin\left(\frac{n\pi}{3}x\right)$

4. $f_e(x) = \begin{cases} -2x+1, & x \in [-3,0] \\ 2x+1, & x \in (0,3] \end{cases}$, $\quad F[f_e](x) = 4 + \sum_{n=1}^{\infty} \frac{12}{n^2\pi^2}\left((-1)^n + 1\right) \cos\left(\frac{n\pi}{3}x\right)$

5. $u(t,x) = \sum_{n=1}^{\infty} \left(-\frac{14}{n\pi}(-1)^n + \frac{2}{n\pi}\right) e^{-\frac{n^2\pi^2}{9}t} \sin\left(\frac{n\pi}{3}x\right)$